16 *Years*

Previous Years Solved Papers

Civil Services Main Examination

(2009-2024)

Mathematics

Paper-I

Topicwise Presentation

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Preface

B. Singh (Ex. IES)

Civil Service is considered as the most prestigious job in India and it has become a preferred destination by all engineers. In order to reach this estimable position every aspirant has to take arduous journey of Civil Services Examination (CSE). Focused approach and strong determination are the pre-requisites for this journey. Besides this, a good book also comes in the list of essential commodity of this odyssey.

I feel extremely glad to launch the fourth edition of such a book which will not only make CSE plain sailing, but also with 100% clarity in concepts.

MADE EASY team has prepared this book with utmost care and thorough study of all previous years papers of CSE. The book aims to provide complete solution to all previous years questions with accuracy.

I would like to acknowledge efforts of entire MADE EASY team who worked day and night to solve previous years papers in a limited time frame and I hope this book will prove to be an essential tool to succeed in competitive exams and my desire to serve student fraternity by providing best study material and quality guidance will get accomplished.

With Best Wishes

B. Singh (Ex. IES)

CMD, MADE EASY Group

Previous Years Solved Papers of

Civil Services Main Examination

Mathematics : Paper-I

CONTENTS

1 Linear Algebra

1. Vector Space Over R and C

1.1 Prove that the set V of the vectors (x_1, x_2, x_3, x_4) in R^4 which satisfy the equations $x_1 + x_2 + 2x_3 + x_4 = 0$ and $2x_1 + 3x_2 - x_3 + x_4 = 0$ is a subspace of R^4. What is the dimension of this subspace. Find one of its bases.

(2009 : 12 Marks)

Solution:

ILD for vertical reaction at A;

$$V = \{(x_1, x_2, x_3, x_4) \in R^4 \mid x_1 + x_2 + 2x_3 + x_4 = 0, 2x_1 + 3x_2 - x_3 + x_4 = 0\}$$

Then clearly $(0, 0, 0, 0) \in V$ and so V is non-empty.

Again let $x = (x_1, x_2, x_3, x_4) \in V$ and $y = (y_1, y_2, y_3, y_4) \in V$, Also let $\alpha, \beta \in R$.

$$\alpha x + \beta y = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3, \alpha x_4 + \beta y_4) \in R^4$$

Also, $\alpha(x_1 + x_2 + 2x_3 + x_4) + \beta(y_1 + y_2 + 2y_3 + y_4) = 0$

$\Rightarrow \alpha x + \beta y$ satisfies $x_1 + x_2 + 2x_3 + x_4 = 0$

Similarly $\alpha x + \beta y$ satisfies 2nd equation as well.

$\therefore \quad \alpha x + \beta y \in V$

Dimension of the Subspace :

Any element of V is a solution to equation

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So, its dimension is same as rowspace of coefficient matrix, i.e., its rank.

Reducing it to row reduced echelon form

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -5 & -1 \end{bmatrix}$$

As it has two non-zero rows in row reduced form.

$$\dim(v) = \text{Rank of matrix} = 2$$

Writing the equation as matrix

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } Ax = 0.$$

So, $\quad V = \{x = (x_1, x_2, x_3, x_4) \in \text{R4} \mid Ax = 0\}$

Clearly $\quad O = (0, 0, 0, 0) \in V$ so V is non-empty.

Let $\quad x = (x_1, x_2, x_3, x_4);\ y = (y_1, y_2, y_3, y_4) \in V$ and $\alpha, \beta \in R$

$$A(\alpha x + by) = A(\alpha x) + A(\beta y)$$

$$= \alpha(Ax) + \beta(Ay) = \alpha\cdot 0 + \beta\cdot 0 = 0$$

$\therefore \quad \alpha x + \beta y \in V$

$\therefore$ V is a vector subspace.

Clearly, V is the null space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & 1 \end{bmatrix}$$

Reducing it to row reduced echelon form

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & 1 \end{bmatrix} \xrightarrow{R_L \leftrightarrow R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -5 & -1 \end{bmatrix}$$

$$\dim(\text{Row space }(A)) = \text{Number of non-zero rows} = 2$$

By Rank-Nullity Theorem :

$$\underset{\text{(Nullity)}}{\dim(\text{Null space})} + \underset{\text{(Rank)}}{\dim(\text{Row space})} = n = 4$$

$$\dim(\text{null space}) = 2$$

$$\therefore \quad \dim(V) = 2$$

For Finding Basis :

$$\dim(\text{null space}) = 2$$

$$\therefore \quad \text{No. of free variables} = 4 - 2 = 2$$

So, we fix 2 variables.

Taking $x_3 = 1$, $x_4 = 0$ first.

$$\left.\begin{aligned} x_1 + x_2 &= -2 \\ 2x_1 + 3x_2 &= 1 \end{aligned}\right\} \begin{aligned} x_1 &= -7 \\ x_2 &= 5 \end{aligned} \qquad x = (-7, 5, 1, 0)$$

Taking $x_3 = 0$, $x_4 = 1$

$$\left.\begin{aligned} x_1 + x_2 &= -1 \\ 2x_1 + 3x_2 &= -1 \end{aligned}\right\} \begin{aligned} x_1 &= -2 \\ x_2 &= 1 \end{aligned} \qquad x = (-2, 1, 0, 1)$$

$\therefore$ $(-7, 5, 1, 0)$ and $(-2, 1, 0, 1)$ are two elements of V. And since they are linearly independent **(because of choice of 3rd and 4th element)** they form a basis.

1.2 **Prove that set V of all 3×3 real symmetric matrices form a linear subspace of the space of all 3×3 real matrices. What is the dimension of this subspace? Find at least one of the bases for V.**

(2009 : 20 Marks)

Solution:

Approach : Use definition of subspaces for first part. For the 2nd impose conditions due to symmetricity on the matrix.

Let V be subset of all 3×3 symmetric matrix.

Then $I_3 \in V$ so V is not empty. Again, let $A, B \in V$.

$\Rightarrow \quad A = A^T$

$B = B^T$ (definition of symmetric)

and $\alpha, \beta \in R$.

Then
$$(\alpha A + \beta B)^T = (\alpha A)^T + (\beta B)^T = \alpha A^T + \beta B^T = \alpha A + \beta B$$

$\therefore$ $\alpha A + \beta B \in V$.

So, V is a vector subspace of the space of all 3×3 real matrices over R.

Dimension : Let
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \in V$$

$\Rightarrow$
$$A^T = A \Rightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

i.e.,
$$b = d,\ c = g,\ h = f$$

$\therefore$
$$A = \begin{bmatrix} a & b & c \\ b & e & f \\ c & f & i \end{bmatrix}$$

Thus, any general element has 6 variables (instead of 9 for a 3×3 real matrix). So, dimension of V is 6.

Basis : Putting each of the variables as 1 and rest 0 gives us a basis, i.e.,

$$B = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$$

1.3 In the n-space R^n, determine whether or not the set $\{e_1 - e_2, e_2 - e_3, \ldots\ldots, e_{n-1} - e_n, e_n - e_1\}$ is linearly independent.

(2010 : 10 Marks)

Solution:

Given the set is $\{e_1 - e_2, e_2 - e_3, \ldots\ldots, e_{n-1} - e_n, e_n - e_1\}$.

Let $a_1(e_1 - e_2) + a_2(e_2 - e_3) + \ldots\ldots + a_{n-1}(e_{n-1} - e_n) + a_n(e_n - e_1) = 0$

$\Rightarrow e_1(a_1 - a_n) + e_2(a_2 - a_1) + e_3(a_3 - a_2) + \ldots\ldots + e_n(a_n - a_{n-1}) = 0$

As $e_1, e_2, \ldots\ldots, e_n$ from basis of R^n, so they are linearly independent.

$\therefore$ $a_1 - a_n = 0$ ...(1)

$a_2 - a_1 = 0$ ...(2)

$a_3 - a_2 = 0$ ...(3)

$a_n - a_{n-1} = 0$ ...(n)

$\therefore$ from eqn. (1) to (n) it can be deducted that

$$a_1 = a_2 = a_3 = \ldots\ldots a_{n-1} = a_n$$

and not need to be zero.

$\therefore$ The given set is linearly dependent.

1.4 Show that the subspaces of $\mathbb{R}^3$ spanned by two sets of vectors {(1, 1, –1), (1, 0, 1) and {(1, 2, –3), (5, 2, 1)} are identical. Also find the dimension of this subspace.

(2011 : 10 Marks)

Solution :

Let W_1 be the subspace generated by the vectors (1, 1, –1), (1, 0, 1).

Consider a matrix A, whose rows are the given vectors (1, 1, –1), (1, 0, 1) and reduce it to row reduced Echelon form.

$\therefore$
$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \text{ by } R_2 \leftrightarrow R_2 - R_1 \quad \text{...(i)}$$

Again W_2 be the subspace generated by the vectors (1, 2, –3), (5, 2, 1).

Consider a matrix B, whose rows are the given vectors (1, 2, –3) and (5, 2, 1) and reduce it to row reduced Echelon form.

i.e.,
$$B = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & -8 & 16 \end{bmatrix} \text{ by } R_2 \to R_2 - 5R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix} R_2 \to \frac{1}{8} R_2$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} R_1 \to R_1 + R_2 \quad \text{...(ii)}$$

From (i) and (ii), we find that the non-zero rows in the row-reduced Echelon forms of matrices A and B are the same.

$\therefore$ Row space of A = Row space of B

$\therefore$ $W_1 = W_2$

Again consider the vectors (1, 1, –1) and (1, 0, 1).

Let $a_1(1, 1, -1) + a_2(1, 0, 1) = (0, 0, 0)$, $a_1, a_2, a_3 \in R$

$\Rightarrow$ $(a_1, a_1, -a_1) + (a_2, 0, a_2) = (0, 0, 0)$

$\Rightarrow$ $(a_1 + a_2, a_1, -a_1 + a_2) = (0, 0, 0)$

$\Rightarrow$ $a_1 + a_2 = 0, a_1 = 0, -a_1 + a_2 = 0$

$\Rightarrow$ $a_1 = 0 = a_2$

$\Rightarrow$ The vectors (1, 1, –1) and (1, 0, 1) are linearly independent.

$\therefore$ The vectors form basis of $W_1 = W_2$.

$\therefore$ Dimension of $W_1 (= W_2) = 2$.

1.5 Prove or disprove the following statement :
If $B = \{b_1, b_2, b_3, b_4, b_5\}$ is a basis for R^5 and V is a two-dimensional subspace of R^5, then V has a basis made of just two members of B.

(2012 : 12 Marks)

Solution :

$$B = \{b_1, b_2, b_3, b_4, b_5\} \text{ is a basis for } R^5.$$

V is a two-dimensional subspace of R^5.

Consider the set $\{b_1, b_2\}$

Let $B' = \{b_1, b_2\}$

Since $B = \{b_1, b_2, b_3, b_4, b_5\}$ is a basis of R^5.

$\Rightarrow b_1, b_2, b_3, b_4, b_5$ are linearly independent and any subset of a L.I. set of vectors is L-I = b_1, b_2 are L.I.

Also, if a basis of vector space contains n elements, then any subset of the vector space having n elements is a basis iff the subset is L.I.

Thus, V will have a basis made of just 2 members of B iff B' is a subset of V.

1.6 Let V be the vector space of all 2 × 2 matrices over the field of real numbers. Let W be the set consisting of all matrices with zero determinant. Is W a subspace of V? Justify your answer.

(2012 : 8 Marks)

Solution:

Let $W_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $|W_1| = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$

and $W_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $|W_2| = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$

Thus, $W_1, W_2 \in W$

Now, $W_1 + W_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and $|W_1 + W_2| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

$\therefore \quad W_1 + W_2 \not\in W$

$\Rightarrow$ W is not a subspace of V as we know that a non-empty subset W of a vector space $V(F)$ is a subspace of V iff W is closed under addition and scalar multiplication.

1.7 Let V be an n-dimensional vector space and $T : V \to U$ be an invertible linear operator. If $\beta = \{X_1, X_2, \ldots, X_n\}$ is a basis of V, show that $\beta' = \{TX_1, TX_2, \ldots, TX_n\}$ is also a basis of U.

(2013 : 8 Marks)

Solution :

Approach : An invertible linear transformation into the same vector space is one-one onto. Anyways linearty of T allows use to prove β' is linearly independent and spans V.

An invertible linear transformation from a vector space to itself is one-one and onto.

$\therefore$ T is one-one and onto.

To prove $\beta' = \{TX_1, \ldots, TX_n\}$ is a basis, we need to show that it spans V and is linearly independent.

Spans V :

Let $x \in V$. Since T is invertible and so one-one there exists $y \in V$ such that

$$T(y) = x$$

Now $y \in V$ and β' is a basis.

So, $\exists\ C_1, C_2, \ldots, C_n \in F$ such that

$$y = C_1X_1 + \ldots + C_nX_n$$

$$T(y) = T(C_1X_1 + \ldots + C_nX_n)$$

$$\Rightarrow \quad x = C_1TX_1 + \ldots + C_nTX_n$$

$\therefore x \in \text{Span}\{TX_1, \ldots, TX_n\}$

β' is linearly independent.

Let it be linearly dependent, then $\exists\ C_1, C_2, \ldots C_n \in F$ not all zero such that

$$C_1TX_1 + \ldots + C_nTX_n = 0$$

$$\Rightarrow \quad T(C_1X_1 + \ldots + C_nX_n) = 0$$

But T is invertible $\Rightarrow$ T is one-one.

$$\therefore \quad \text{Kernel } T = \{0\}$$

$$\Rightarrow \quad C_1X_1 + \ldots + C_nX_n = 0$$

with not all of C_i's zero.

But this means $X_1, X_2, \ldots, X_n$ are not linearly independent, a contradiction.

$\therefore$ β' is linearly independent set and thus β' is a basis.

1.8 Show that the vector $X_1 = (1, 1 + i, i)$, $X_2 = (i, -i, 1 - i)$ and $X_3 = (0, 1 - 2i, 2 - i)$ in C_3 are linearly independent over the field of real numbers but are linearly dependent over the field of complex numbers.

(2013 : 8 Marks)

Solution :

Approach : Linear independence depends on the constants of the scalar fields over which the vector space is defined. For real constants the linear combination can not be made 0. But it can be for complex constants.

Let X_1, X_2, X_3 be three vectors over R.

Let $C_1, C_2, C_3 \in R$ such that

$$C_1X_1 + C_2X_2 + C_3X_3 = 0$$

$$\Rightarrow \quad C_1(1, 1 + i, i) + C_2(i, -i, 1 - i) + C_3(0, 1, -2i, 2 - i) = 0$$

$$\Rightarrow \quad [C_1 + C_2 i, (C_1 + C_3) + (C_1 - C_2 - 2C_3)i, C_2 + 2C_3 + (C_1 - C_2 - C_3)i] = 0$$

$$\Rightarrow \quad C_1 + C_2 i = 0$$

$$\Rightarrow \quad (C_1 + C_3) + (C_1 - C_2 - 2C_3)i = 0 \quad \text{(using first two terms)}$$

$$\Rightarrow \quad C_1 = 0,\ C_2 = 0,\ C_3 = 0$$

So, X_1, X_2, X_3 are linearly independent over R.

Linear dependence on C.

Let $i, -1, 1 \in C$ be the constants.

$$= i(1, 1 + i, i) + (-1)(i, -i, 1 - i) + 1(0, 1 - 2i, 2 - i)$$

$$= (i, i - 1, -1) + (-i, i, -1 + i) + (0, 1 - 2i, 2 - i)$$

$$= (0, 0, 0)$$

$\therefore X_1, X_2, X_3$ are linearly dependent on C.

1.9 Find one vector in R^3 which generates the intersection of V and W, where V is the xy-plane and w is the space generated by the vectors (1, 2, 3) and (1, –1, 1).

(2014 : 10 Marks)

Solution:

Let $R^3 = \{(x,y,z) / x,y,z \in IR\}$ be the given vector space.

Given that V is the xy-plane.

$$\therefore \quad V = \left\{\frac{(x,y,z)\in IR^3}{z=0 : x,y\in IR}\right\} \quad \text{...(i)}$$

and given that w is the space generated by the vectors (1, 2, 3) and (1, –1, 1).

For this, we find a homogeneous system whose solution set W is generated by

$$S = \{(1, 2, 3), (1, -1, 1)\}$$

$$W = \{\alpha(1, 2, 3) + \beta(1, -1, 1); \alpha, \beta(-1)\}$$

$$= \{(\alpha + \beta, 2\alpha - \beta, 3\alpha + \beta), \alpha, \beta \in IR)\}$$

$$V = \{(x, y, 0); \lambda, y \in IR\}$$

Now for $V \cap W$

$$x = \alpha + \beta,\ y = 2\alpha - \beta \text{ and } 3\alpha + \beta = 0$$

$$\Rightarrow \quad \beta = -3\alpha$$

$$\Rightarrow \quad x = -2\alpha,\ y = 5\alpha \text{ and } z = 0$$

$$\therefore \quad V \cap W = \{-2\alpha, 5\alpha, 0; \alpha \in IR\}$$

Clearly $V \cap W$ is spanned by $(-2, 5, 0) \in IR^3$.

1.10 Let v and w be the following subspaces of R^4.

$$v = \{(a, b, c, d) : b - 2c + d = 0\}$$

and

$$w = \{(a, b, c, d) : a = d, b = 2c\}$$

Find a basis and the dimension of (i) v (ii) w (iii) $v \cap w$.

(2014 : 15 Marks)

Solution:

We observe that (a, b, c, d) G

$$\Rightarrow \quad b - 2c + d = 0$$

$$\Rightarrow \quad (a, b, c, d) = (a, 5, c, 2, cb)$$
$$= (a, 0, 0, 0) + (0, b, 0, -b) + (0, 0, c, 2c)$$
$$= a(1, 0, 0, 0) + b(0, 1, 0, -1) + c(0, 0, 1, 2)$$

This shows that every vector in v is a linear combination of the three linearly independent vectors (1, 0, 0, 0), (0, 1, 0, –1) (0, 0, 1, 2)

Thus, a basis of v is

$$A = \{(1, 0, 0, 0), (0, 1, 0, -1), (0, 0, 1, 2)\}$$

Hence, **dim v = 3**

Now, $(a, b, c, d) \in w \Rightarrow \quad a = d, b = 2c$

$$\Rightarrow \quad (a, b, c, d) = (a, 2c, c, a) = (a, 0, 0, a) + (0, 2c, c, 0)$$
$$= a(1, 0, 0, 1) + c(0, 2, 1, 0)$$

which shows that w is generated by the linearly independent set {(1,0,0,1), (0,2,1,0)}

$\therefore$ A basis for w is $= \{(1, 0, 0, 1), (0, 2, 1, 0)\}$

and $\dim w = 2$

$(a, b, c, d)\ v \cap w \Rightarrow \quad (a, b, c, d) \in v$ and (a, b, c, d)

$$\Rightarrow \quad b - 2c + d = 0, a = d, b = 2c$$
$$\Rightarrow \quad (a, b, c, d) = (0, 2c, c, 0) = c(0, 2, 1, 0)$$

Hence a basis of $v \cap w$ is {0, 2, 1, 0}

and $\dim (v \cap w) = 1$

1.11 The vectors $v_1 = (1, 1, 2, 4)$, $v_2 = (2, -1, -5, 2)$, $v_3 = (1, -1, -4, 0)$ and $v_4\ (2, 1, 1, 6)$ are linearly independent. Is it true? Justify your answer.

(2015 : 10 Marks)

Solution:

We form a matrix with these given vectors as rows and reduce it to row-echelon form to investigate its rank.

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 - \frac{2}{3}R_2 \\ R_4 \to R_4 - \frac{R_2}{3} \end{matrix}$$

Since rank (A) = 2 < Number of rows (vectors)

Hence, the vectors are not linearly independent.

1.12 Find the dimension of the subspace of R^4, spanned by the set {(1,0,0,0), (0,1,0,0), (1,2,0,1), (0,0,0,1)}. Hence, find its basis.

(2015 : 12 Marks)

Solution :

We find the echelon form of the matrix formed by given vectors taking as rows.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_4 \to R_4 - R_3$$

Hence, Rank of Matrix = 3

∴ Dimension of subspace = 3

For the basis, we take the vectors from the original matrix which correspond the non-zero rows in the echelon form.

∴ Basis = {(1,0,0,0), (0,1,0,0), (1,2,0,1)}

1.13 If

$$W_1 = \{(x, y, z) : x + y - z = 0\}$$
$$W_2 = \{(x, y, z) : 3x + y - 2z = 0\}$$
$$W_3 = \{(x, y, z) : x - 7y + 3z = 0\}$$

then find $\dim(W_1 \cap W_2 \cap W_3)$ and $\dim(W_1 + W_2)$.

(2016 : 3 Marks)

Solution :

Let $(x, y, z) \in W_1 \cap W_2 \cap W_3$

$$\therefore \quad x + y - z = 0 \quad \text{...(i)}$$
$$3x + y - 2z = 0 \quad \text{...(ii)}$$
$$x - 7y + 3z = 0 \quad \text{...(iii)}$$
$$\Rightarrow \quad x + y - z = 0$$
$$\Rightarrow \quad x + y - z = 0$$
$$-2y + z = 0$$
$$\Rightarrow \quad -2y + z = 0$$
$$\therefore \quad y = \frac{z}{2}$$
$$x = \frac{z}{2}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z/2 \\ z/2 \\ z \end{bmatrix} = z\begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

∴ $\dim(W_1 \cap W_2 \cap W_3) = 1$

Again $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$ ...(*)

$\dim(W_1 \cap W_2) \Rightarrow$ $x + y - z = 0$

$$3x + y - 2z = 0$$
$$\Rightarrow \quad 2x - z = 0$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z/2 \\ z/2 \\ z \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix} z$$

∴ $\dim(W_1 \cap W_2) = 1$

(∗) gives $\dim(W_1 + W_2) = 2 + 2 - 1 = 3$

1.14 Suppose U and W are distinct four dimensional subspaces of a vector space V, where dim $V = 6$. Find the possible dimensions of subspace $U \cap W$.

(2017 : 10 Marks)

Solution:

$$\dim(U) = \dim(W) = 4$$

U and W are subspaces of V.

We know, $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$

∴ $\dim(U \cap W) = \dim(U) + \dim(W) - \dim(U + W)$

$= 8 - \dim(U + W)$ ...(i)

Now, $U + W$ is a subspaces of V while U and W are subspaces of $U + W$.

∴ Possible $\dim(U + W) = 4, 5, 6$

Hence, by (3)

Possible $\dim(U \cap W) = 4, 3, 2$

But $\dim(U \cap W)$ cannot be 4 otherwise $U \cap W = U = W$, but U and W are distinct.

∴ $\dim(U \cap W) = 2$ or 3

1.15 Express basic vectors $e_1 = (1, 0)$ and e_2 $(0, 1)$ as combinations of $\alpha_1 = (2, -1)$ and $\alpha_2 = (1, 3)$.

(2018 : 10 Marks)

Solution:

Let any vector, say $(x, y) = a\alpha_1 + b\alpha_2$

∴ $(x, y) = (2a + b, -a + 3b)$

Comparing LHS and RHS, we get

$$a = \frac{3x - y}{7}, \; b = \frac{x + 2y}{7}$$

For $e_1 = (1, 0) : x = 1, y = 0$

∴ $a = \frac{3}{7}, \; b = \frac{1}{7}$

So, $e_1 = \frac{3}{7}\alpha_1 + \frac{1}{7}\alpha_2$

For $e_2 = (0, 1) : x = 0, y = 1$

∴ $a = -\frac{1}{7}, \; b = \frac{2}{7}$

So, $e_2 = -\frac{1}{7}\alpha_1 + \frac{2}{7}\alpha_2$

∴ $$e_1 = \frac{3\alpha_1 + \alpha_2}{7}, \; e_2 = \frac{-\alpha_1 + 2\alpha_2}{7}$$

1.16 Consider the set V of all $n \times n$ real magic squares. Show that V is a vector space over R. Give examples of two distinct 2×2 magic squares.

(2020 : 10 Marks)

Solution:

Let V be the set of all $n \times n$ real magic squares. Let $\alpha, \beta \in V$, and sum of all rows and columns of α and β be k_1 and k_2. Again sum of rows and column of matrix $\alpha + \beta$ will be $k_1 + k_2 \Rightarrow \alpha + \beta \in V$ (Internal composition).

Now, let $a \in R$

Then $a\alpha \in V$ as the sum of rows and columns of $a\alpha$ will be ak_1. Thus external composition is satisfied.

$\Rightarrow$ (V, +) forms an abelian group.

(i) $\alpha + \beta \in V$ for $\alpha, \beta, \gamma \in V$

(ii) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

(iii) Zero matrix is identity

(iv) $-\alpha$ is inverse of α matrix

(v) Commutative property $\alpha + \beta = \beta + \alpha$. Again for $a, b \in R$, $\alpha, \beta \in V$, we have $a(\alpha + \beta) = a\alpha + b\beta$ (operation in matrix)

(vi) $(a + b)\alpha = a\alpha + b\beta$

(vii) $(ab)\alpha = a(b\alpha)$

(viii) $1 \cdot \alpha = \alpha$ (multiplication with identity matrix). Hence, V is a vector space. For examples: $\alpha = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$.

1.17 Prove that any set of n linearly independent vectors in a vector space V of dimension n constitutes *a* basis for V.

(2022 : 10 marks)

Solution:

Given Dim $V = n$

Let $S = \{a_1, a_2, a_3 \ldots a_n\}$ be linearly independent (L.I.) set of n vectors in V.

If S is not a basis of V then it can be extended to form a basis of V.

In this case, the basis will contain more than n vectors.

But every basis of V must contain exactly n vectors.

Therefore our presumption is wrong and S must be basis of V.

1.18 Let the set $P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{matrix} x - y - z = 0 \text{ and} \\ 2x - y + z = 0 \end{matrix} \right\}$ be the collection of vectors of a vector space $R^3(R)$. Then

(i) prove that P is a subspace of R^3.

(ii) find a basis and dimension of P

(2022 : 10 + 10 marks)

Solution:

$$\text{Given } P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x - y - z = 0 \text{ and } 2x - y + z = 0 \right\}$$

(i) We have $x - y - z = 0$

then $x = y + z$ ...(i)

and $2x - y + z = 0$, thus the value of $x = y + z$ then we have

$2(y + z) - y + z = 0$

$\Rightarrow$ $2y + 2z - y + z = 0$

$\Rightarrow$ $y + 3z = 0 \Rightarrow y = -3z$

then $x = -3z + z = -2z$

Thus, we have $(x, y, z) = (-2z, -3z, z) = z(-2, -3, 1)$

So, we have $P = \{z(-2, -3, 1) \text{ for } z \in R\}$

To prove, P is the vector space of $R^3(R)$

Let $\alpha_1 = z_1(-2, -3, 1)$ and $\alpha_2 = z_2(-2, -3, 1)$

belongs to P then $a\alpha_1 + b\alpha_2 \in P$

where $\alpha_1, \alpha_2 \in P$ and $a, b \in R$

we have, $a\{z_1(-2, -3, 1)\} + b\{z_2(-2, -3, 1)\}$

$= \{-2(az_1 + bz_2), -3(az_1 + bz_2), (az_1 + bz_2)\} \in P$ as $az_1 + bz_2 \in R$

Thus P is the vector space of $R^3(R)$. Hence proved.

(ii) As we know $P = \{z(-2, -3, 1) \text{ for all } z \in R\}$

Thus Dim(P) = 1

and Basis set of $P = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$

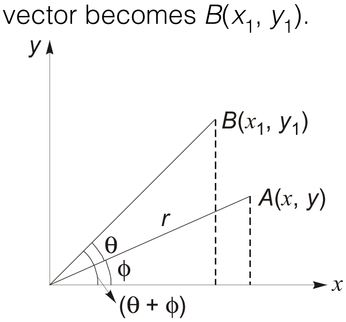
1.19 Find a linear map T. $R^2 \to R^2$ which rotates each vector of R^2 by an angle θ. Also, prove that for $\theta = \frac{\pi}{2}$, T has no eigenvalue in R.

(2022 : 15 marks)

Solution:

Let $A(x, y)$ is the initial vector of R^2 and after rotation by an angle θ the final vector becomes $B(x_1, y_1)$.

Here, $x = r\cos\phi$, $y = r\sin\phi$ ...(i)



$\therefore$ $(x, y) = (r\cos\phi, r\sin\phi)$

The final vector (x_1, y_1) after a rotation by angle θ becomes

$$x_1 = r\cos(\theta + \phi)$$
$$= r[\cos\theta\cos\phi - \sin\phi\sin\theta]$$
$$= r\cos\theta\cos\phi - r\sin\phi\sin\theta$$

Put the value of x and y from eq.(i), we have

$$x_1 = x\cos\theta - y\sin\theta$$

Now, calculate y_1, we have

$$y_1 = r\sin(\theta + \phi)$$
$$= r[\sin\theta\cos\phi + \cos\theta\sin\phi) = r\sin\theta\cos\phi + r\cos\theta\sin\phi$$